A Brief Overview of Random Walks and Electrical Networks

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Abstract

This paper gives an introductory overview of the concept of both random walks and electrical networks in graph theory using only undergraduate level mathematics. Random walks are stochastic processes that describe paths consisting of series of random steps in mathematical spaces. Electrical networks are intuitively interconnections of electrical components such as resistors, capacitors, and inductors in a circuit, modeled using current, voltage, and resistance; although in this paper we focus on electrical networks in the context of random walks. This paper covers fundamental definitions, history, rudimentary examples, and important theorems in both random walks and electrical networks and analyze the relationship between them.

1 Introduction

Given a graph G and starting vertex v, select an adjacent vertex uniformly and randomly among the neighbors, and move to this vertex. The sequence of random vertices selected using this method defined over a set of states and a matrix of probabilities is called a *Markov chain*, and the sequence would be considered as a random walk on the graph G. Random walks are common in algorithm design and probabilistic analysis and also have numerous other applications.

One important application is electrical networks or circuits that can be represented as a graph $G(V, E)$ or vice versa. If we treat each edge $xy \in E(G)$ that connects vertices, there is defined resistance r_{xy} for the edge and there is a current i_{xy} that flows through the wire. Common circuit laws such as Kirchhoff's Law and Ohm's Law also applies here as underlying properties of electrical networks in the context of random walks.

Both random walks and electrical networks have physics as the source for the problems represented using expert level mathematical abstraction that are extremely difficult to understand and appreciate. However, with only college level mathematics, we can still analyze the relationship between electrical networks and random walks. This paper will give a brief overview of each concept using basic concepts in graph theory while avoiding graduate research level mathematics concepts and jargon.

2 Basic Definitions

These are some very basic definitions that are the building blocks for our discussions regarding random walks and electrical networks that we have not yet touched upon in the MATH 4710 class. This mostly only includes terms that are fundamental to deeper definitions and theorems, which will be addressed separately and defined later during the in-depth mathematics discussion. Some of these terms (i.e. current, voltage, resistance) might be present or can even be considered as elementary in other disciplines, but for the context of graph theory, they are still listed here for completeness purpose.

Definition 2.1. A **stochastic process** is a mathematical phenomena that has a random probability distribution statistically, although it cannot be precisely predicted. Coin flipping and random walks are both examples of various categories of stochastic processes with different mathematical properties [1].

Definition 2.2. A **random walk**, or a drunkard's walk colloquially, is defined as a stochastic process or a random process, which describes a path consisting of a series of random steps in a mathematical space (such as the integer space). It is one of the first chance process that is studied in probability theory, and has many applications [1].

Definition 2.3. The **Law of Large Numbers (LLN)** is a theorem in probability theory that explains the results when the same experiment is performed many times. According to LLN, the average of the results of many trials should be close to the expected value and tends to approach the expected value as more trials are performed. LLN is important because it ensures stability in the long-term results of random events after a large number of iterations [3].

An example of LLN in practice would be the task of distinguishing between a probabilistic random event and a non-random event. In the case of a rigged coin flip, flipping the coin once will offer little to no information at all about the state of the coin. However, flipping the coin $k = 10$ times will give us 99.9% certainty about the state of the coin since the probability that a coin will land on the same side 10 times in a row is $(0.5)^{10}$ and $1 - (0.5)^{10} = 0.999$. Note that for the sake of this example we assume that the rigged coin will always land on the same side with a hundred percent certainty.

Definition 2.4. A **Markov chain**, or a Markov process, is a stochastic model that describes a series of possible events, and the probability of each event depends on the state reached in the previous event [1].

Definition 2.5. An **electrical network** is an interconnection of electrical components. Some common electrical components include resistors, capacitors, inductors, transistors, and they are modeled using definitions such current, voltage, resistance, capacitance, or inductance.

Here are some basic yet important terms used in electrical networks that represent how they behave physically in the real world. Most if not all of the equations used in electrical networks revolves around some of these terms and concepts.

Definition 2.6. The **current** is the rate of electrical charge flows, commonly measured in ampere.

Definition 2.7. The **voltage** is the potential difference in charge between two points in the electrical network, commonly measured in volt.

Definition 2.8. The **resistance** is the measure of an object's resistance to electrical flow, commonly measured in ohm.

3 History

The term random walk was first introduced by Karl Pearson in *Nature*, a journal article published in 1905 [7]. He proposed a hypothetical problem of a man walking l yards in one direction, before turning randomly and walking another l yards in another direction. This process is repeated n times. He said that this problem interested him considerably, and he asked the readers whether there already exists an integrated solution to the problem he proposed that calculates the probability that the man is at a position distance r away from his starting points expressed in terms of l/n [7].

However, the first person that took this concept seriously in a research context was Russian mathematician Andrey Markov, who studied the ideas about chains of linked probabilities. The original intellectual thread came all the way from the famous Swiss mathematician Jacob Bernoulli. Bernoulli stated in *Ars Conjectandi* that if you keep flipping a fair coin, the number of heads will approach the number of tails as the number of flips goes to infinity [2]. This is what eventually became known as the Law of Large Numbers (LLN). Although this concept seems intuitive, it is difficult to find a rigorous proof that explains the precise reasoning behind this concept. Bernoulli attempted a version of a proof, followed by Pafnuty Chebshev, another Russian mathematician who published a broader version of a proof, before being further refined by Markov [2].

Andrey Markov also studied Alexander Pushkin's novel *Eugene Onegin* in verses and sorted through all the patterns of how vowels and consonants are used in the poetry [4]. Eventually, he summarized and published his findings to the Imperial Academy of Sciences in St. Petersburg in January 23, 1913 [3]. Although from a modern linguistic point of view, he did not fully consider all the choices of poem structure such as rhyme and length, but rather treated the text as a stream of letters, and realized that the letter probabilities are not independent by a large margin, and depends on the adjacent letters [4]. His analysis and findings not only altered the understanding of poems, but also developed a new technique that extended into a new branch of probability theory [3]. This technique is now known as the Markov chain, as explained earlier in the definition section [3].

Although Karl Pearson's original problem was not reportedly answered or solved as far as I know, his idea was addressed by George Pólya in 1921, a Hungarian mathematician that proposed the famous Pólya's theorem, that a random walker on an infinite street network in a d-dimensional space is bound to return to the stating point when $d = 2$, but has a positive probability of approaching infinity and not return to the stating point when $d \geq 3$ [6]. This seemingly intuitive theorem is actually quite complicated and requires techniques from classical electrical network theory to prove rigorously [6]. The proof will be explored later in the paper and is an important part of random walks on infinite networks, although at the time he called it "street networks" instead.

Furthermore, although the origin of the idea of the connection between random walks and electric networks are unknown, the topic has been extensively recognized and explored by many mathematician. However, the first person to apply Rayleigh's method of dimensional analysis, a very common conceptual tool used in physics and engineering that will be explained later, to random walks seems to be the British mathematician Crispin St. John Alvah Nash-Williams in 1959, although the American mathematician Halsey Royden had applied Rayleigh's method of dimensional analysis to a similar and potentially equivalent problem dealing with harmonic functions on open Riemann surfaces earlier than Nash-Williams in 1952 [5, 8].

4 Rudimentary Random Walks

The most rudimentary example for a randoms walk is the example of a man standing at a point x, he can either walk one block to the right, or walk one block to the left, and there is an equal probability 1/2 of either happening. This is the most intuitive representation of the random walk problem and is quite similar to Karl Pearson's original question, which was

2-dimensional with four possible directions instead of two. The problem for the example would be to determine the probability that the man is a position 0 on the line before reaching a distinct position y after an arbitrary number of steps. The solution for this rudimentary problem turns out to be simple as well, and the probability $p(x)$ is strictly based on the distance between starting point x divided by the distance between x and y. Although this is a very limited rudimentary example for a 1-dimensional space, which is just a straight line [1].

5 Rudimentary Electrical Networks

The concept of random walks can be considered in the context of electrical networks. The rudimentary problem for random walks can be transformed into an electrical network with a voltage established across the ends, with one end being the ground. In addition, equal resistors are connected in series such that there is a resistor between our conceptual "points" across the electrical network. By Ohm's Law, the current $i_x y$ flowing point x to point y is $\frac{v(x)-v(y)}{R_{xy}}$ with $v(x)$ and $v(y)$ being the voltage defined at each point, respectively, and $r_x y$ as the sum of the resistance series from point x to point y. By Kirchhoff's Law, we know that for points along the electrical network line, $\frac{v(x-1)-v(x)}{R} + \frac{v(x+1)-v(x)}{R} = 0$ with R being the magnitude of the corresponding resistance. We can solve the above equation and cancel the R and get $v(x) = \frac{v(x+1)+v(x-1)}{2}$. This then represents the voltage at x, which is an equivalent equation to the probability $p(x)$ mentioned in one-dimensional random walks applied on the electrical network [1].

6 Random Walks on General Electrical Networks using Markov Chains

In order to further generalize and understand the idea of random walks on higher dimensional networks, we attempt to analyze random walks on general resistor networks using Markov chains, which we have defined earlier in the definition section.

Definition 6.1. The **conductance** is the ability for electrical charge to flow, it is conveniently defined as $C = 1/R$, and is measured in siemens.

If G is a connected graph, then each edge xy in G has a resistance, we name it R_{xy}. The conductance for each edge would be the inverse of the resistance, which is $C_{xy} = 1/R_{xy}$. With this new definition of conductance, the *random walk* is more formally defined as a Markov chain with the transition matrix $\mathbf{P}$ given by $P_{xy} = C_{xy}/C_x$ with C_x being the sum of all conductance from node x to all other node $\Sigma_y C_{xy}$. The Markov chain matrix

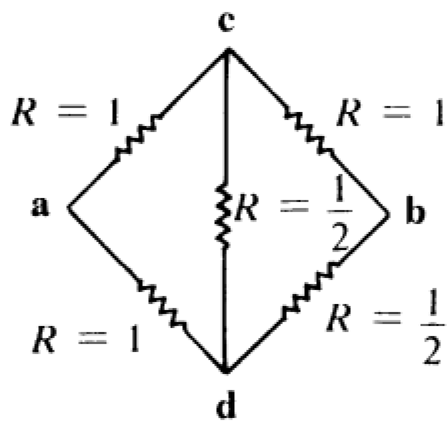

Figure 1: An example resistance network

generated will represent the unique probability between two points in the fixed electrical network.

An example electrical network with marked resistance is shown in Figure 1, the represents a very standard and commonly seen simple circuits generally studied in electronics [1]. We can use our defined definition for Markov chain to create the following matrix $\mathbf{P}$:

$$\begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/5 & 2/5 & 2/5 & 0 \end{bmatrix}$$

With the graph G is being connected, the walker can randomly travel between any two states using the above Markov chain matrix, and the probability is well-defined based on the resistance and conductance of the electrical network. A graphical representation created by transforming the Markov chain into a directed graph is shown in Figure 2 [1]. Although the examples we have considered so far are simple and regular electrical circuits, the basic concept for the Markov chain model can be adapted and applied to more complicated electrical network.

7 Pólya's Theorem

Another important concept in the realms of random walks is the aforementioned Pólya's Theorem, proposed by George Pólya in 1921. Pólya invested random walks and very structured infinite graphs as shown Figure 3 on various dimensions [1]. These graphs are also called *lattices* in this context by Pólya and they are constructed by joining vertices with

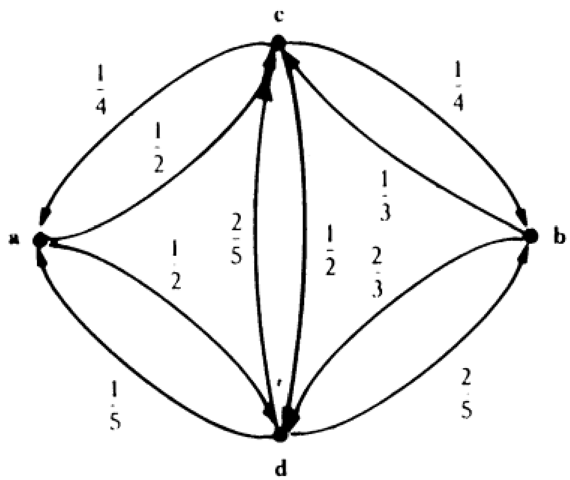

Figure 2: The Markov chain for the electrical network represented as a directed graph

integer coordinates using undirected line segment to their respective nearest neighbors, the connecting segments represent edges in the graph [1].

As shown in the graph above, when $d = 1$, the lattice is just an infinite line divided into even segments with length equals to one, when $d = 1$, the lattice represents an infinite network of streets and blocks, and when $d = 3$, the lattice is much more complicated and resembles a jungle gym [1].

Pólya proposed the question, if the probability of moving into any direction in the lattice is equal to each otherwise, is it guaranteed that the walk will return to its starting point [1]?

Definition 7.1. If the walk is guaranteed to return to its starting point, we call the walk **recurrent**. If there is a positive probability that the point will never return to its starting point, we call the walk **transient** [6].

Pólya was able to answer the question and proved the following theorem in his 1921 work *Uber eine Aufgabe betreffend die Irrfahrt im Strassennetz*, and this theorem lays the groundwork for our understanding of random walks and its many applications [6].

Theorem 7.2 (Pólya's Theorem)**.** *Simple random walk on a d-dimensional lattice is recurrent for* $d = 1, 2$ *and transient for* $d >= 3$.

Proof. Unfortunately, the full detail of the proof is incredibly complicated and even the simplified version proposed by modern researchers requires at least 1000 words to fully explain the idea, with the original proof proposed by Pólya being even longer, sitting at

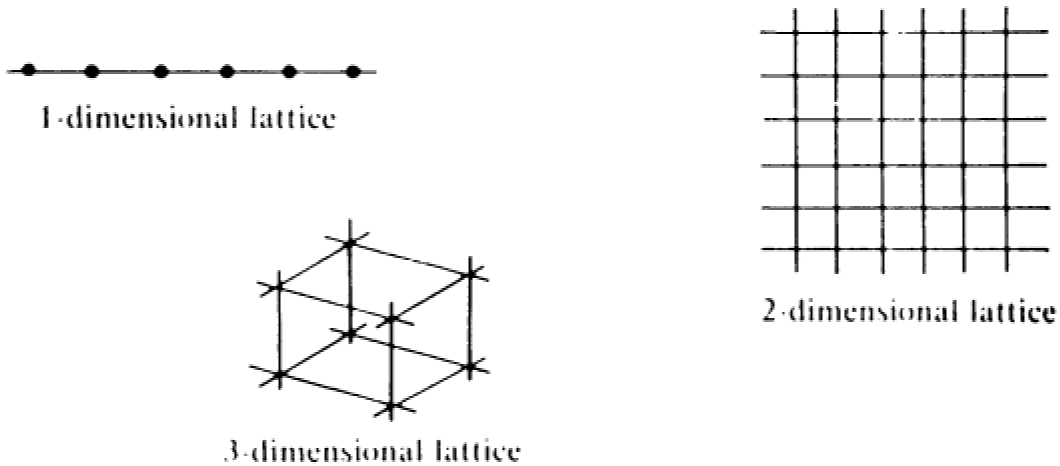

Figure 3: 1-dimensional lattice, 2-dimensional lattice, and 3-dimensional lattice

12 pages long and is written in German. Hence we will briefly run through the idea of the proof, which is done using Rayleigh's method as a tool [1, 6].

For $d = 1$, it is very easy to prove because a simple random walk is recurrent will mean that the network has infinite resistance, and vice versa. This is obvious because an infinite line of unit resistors with 1-ohm resistance obviously has infinite resistance cumulatively [1].

For higher dimensions, we also need to decide if the d-dimensional lattice has infinite resistance, and we use Rayleigh's method as a tool to arrive at such decision [1].

Law 7.3 (Monotonicity Law)**.** The effective resistance between two specified nodes is monotonic in the branch resistances [1].

This law is generally broken down into two smaller procedures that are used a shortcut for proving many theorems including Pólya's Theorem.

Law 7.4 (Shorting Law)**.** Shorting certain sets of nodes can only reduce the effective resistance between the two nodes in the electrical network [1].

Law 7.5 (Cutting Law)**.** Cutting certain branches can only increase the effective resistance between the two nodes in the electrical network [1].

First, Rayleigh's idea was to use the Shorting Law and the Cutting Law to get lower and upper bound for the resistance. With the Shorting Law, we can short nodes that forms squares in the 2-dimensional lattice using Rayleigh's Shorting Law. We soon realize that the resistance of the network will lead to infinity because the modified network after shorting leads to infinity. This means that when $d = 2$, the walk is recurrent [1].

For $d = 3$, we delete branches of network using Rayleigh's Cutting Law such that the residual network has finite resistance. This is slightly more difficult to do, we had to try to find embedding trees before eventually arriving at the conclusion that the escape probability is positive and the walk is transient [1].

There are many more ways to prove Pólya's Theorem as it turns out, although most other proofs required complex mathematical concepts to prove [1].

□

8 Sub Areas

There really is not any noteworthy subarea for the concept of random walks and electrical networks as they are the building blocks for many other scientific fields. However, there are a lot of specific applications using random walks in many different fields of study. In biology, genetic drift in the population of a specie is based on the concept of random walks. In physics, the Brownian motion exhibited by the random movement of molecules in liquids and gases are also modeled by random walks. Moreover, random walks can also be used to model how a person makes decision based on the available options in general psychology.

9 Summary

Overall, both random walks and their applications within the context of electrical networks are concepts observed and derived from real world behaviors of physical phenomenon, and are represented partially using some relevant graph theory concepts that we have learned in the MATH 4710 course such as walks, paths, directed graphs, and network flows, as discussed in the paper. However, it does seem like both randoms walks and electrical networks involve techniques and understandings from other branches of mathematics and physics such as probability theory and complex electrical laws applied over the linear resistant networks such as Kirchhoff's Law and Ohm's Law, both of which are very common techniques seen in electrical engineering, although extremely uncommon in graph theory.

Nevertheless, the concept of random walks have a large number of applications in many areas of the scientific and engineering field as it explains the observed behaviors of many stochastic processes as a fundamental model. Thanks to the basic building block of graph theory constructs and its principal properties and theories, we are able to represent and define this conceptually important idea sufficiently using only undergraduate level mathematics.

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